Math 247A Lecture 23 Notes

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1 Introduction to Oscillatory Integrals

1.1 Decay of integrals with compactly supported integrand

Oscillatory integrals are of two types

1. First kind:
$$I(\lambda) = \int e^{i\lambda\phi(x)}\psi(x)\, dx,$$
where $\lambda > 0$, $\phi : \mathbb{R}^d \to \mathbb{R}$, and $\psi : \mathbb{R}^d \to \mathbb{C}$. In this case, we are interested in the asymptotic behavior of $I(\lambda)$ as $\lambda \to \infty$ (think of λ as time). This is covered in Chapter 8 of Stein's Harmonic Analysis textbook.

2. Second kind:
$$(T_\lambda f)(x) = \int e^{i\lambda\phi(x,y)} K(x, y) f(y)\, dy,$$
where $\lambda > 0$, $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, and $f : \mathbb{R}^d \to \mathbb{C}$. We are interested in the asymptotic behavior of the norm of T_λ as $\lambda \to \infty$. This is covered in Chapter 9 of Stein's Harmonic Analysis textbook.

In this class, we'll discuss oscillatory integrals of the first kind, first for $d = 1$, where we will develop a more complete theory.

Proposition 1.1. *Let $\phi : \mathbb{R} \to \mathbb{R}$, $\psi : \mathbb{R} \to \mathbb{C}$ be smooth functions. Assume* $\operatorname{supp} \psi \subseteq (a, b)$ *(nonempty interval), and suppose $\phi'(x) \neq 0$ for all $x \in [a, b]$. Then $I(\lambda) = \int_a^b e^{i\lambda\phi(x)}\psi(x)\, dx$ satisfies $|I(\lambda)| \lesssim_N \lambda^{-N}$ for all $N \geq 0$.*

Proof. We use integration by parts. Write
$$e^{i\lambda\phi(x)} = \frac{1}{i\lambda\phi'(x)} \frac{d}{dx}(e^{i\lambda\phi(x)}).$$

Integrating by parts,

$$I(\lambda) = \int_a^b \frac{d}{dx}(e^{i\lambda\phi(x)}) \left(\frac{\psi(x)}{i\lambda\phi'(x)}\right) dx = \int_a^b e^{i\lambda\phi(x)} \left[-\frac{d}{dx}\frac{\psi(x)}{i\lambda\phi'(x)}\right] dx.$$

Let

$$(Df)(x) = \frac{1}{i\lambda\phi'(x)}\frac{df}{dx}(x).$$

The transpose of D is

$${}^tDf(x) = -\frac{d}{dx}\left(\frac{f(x)}{i\lambda\phi'(x)}\right).$$

Note that

$$D^N(e^{-\lambda\phi}) = e^{i\lambda\phi} \qquad \forall N \geq 1.$$

Now

$$\begin{aligned} I(\lambda) &= \int_a^b D^N(e^{i\lambda\phi(x)})\psi(x)\,dx \\ &= \int_a^b e^{i\lambda\phi(x)}({}^tD)^N\psi(x)\,dx \\ &= \int_a^b e^{i\lambda\phi(x)}\left[-\frac{d}{dx}\frac{1}{i\lambda\phi'(x)}\right]^N. \end{aligned}$$

We get

$$|I(\lambda)| \lesssim \lambda^{-N}\sum_{k=0}^{N}\sum_{\substack{\beta+\alpha_1+\cdots+\alpha_k=N\\ \alpha_j\geq 1}}\left\|\frac{\psi^{(\beta)}\phi^{(1+\alpha_1)}\cdots\phi^{(1+\alpha_k)}}{(\phi')^{N+k}}\right\|_{L^1(a,b)} \lesssim_N \lambda^{-N}. \qquad \square$$

Remark 1.1. If ψ is not compactly supported inside (a,b), then we don't expect better than λ^{-1} decay.

$$\int_a^b e^{i\lambda x}\,dx = \frac{e^{i\lambda b}-e^{i\lambda a}}{i\lambda}.$$

1.2 The Van der Corput lemma

Proposition 1.2 (Van der Corput lemma)**.** *Let $\phi : \mathbb{R} \to \mathbb{R}$ be smooth. Fix $k \geq 1$, and assume $|\phi^{(k)}(x)| \geq 1$ for all $x \in [a,b]$. If $k = 1$, assume also that ϕ' is monotonic on $[a,b]$. Then $I(\lambda) = \int_a^b e^{i\lambda\phi(x)}\,dx$ satisfies $|I(\lambda)| \leq c_k\lambda^{-1/k}$, where c_k is independent of ϕ, λ, a, b.*

Remark 1.2. If $k = 1$, the assumption $|\phi'(x)| \geq 1$ is not sufficient to get the claim. Set $\lambda = 1$. Then

$$|I(\lambda)| = \left|\int_a^b e^{i\phi(x)}\,dx\right| \geq \left|\int_a^b \cos(\phi(x))\,dx\right|.$$

If ϕ' is large on $\{x : \cos(\phi(x)) < 0\}$ and small on $\{x : \cos(\phi(x)) > 0\}$, then $|\{x : \cos(\phi(x)) < 0\}| \ll |\{x : \cos(\phi(x)) > 0\}|$. In particular, $|I(\lambda)| \xrightarrow{b\to\infty} \infty$.

Let's prove the Van der Corput lemma.

Proof. We argue by induction on k. First, let $k = 1$. Then

$$\begin{aligned} I(\lambda) &= \int_a^b e^{i\lambda\phi(x)}\,dx \\ &= \int_a^b \frac{1}{i\lambda\phi'(x)} \frac{d}{dx}(e^{i\lambda\phi(x)})\,dx \\ &= \frac{e^{i\lambda\phi(b)}}{i\lambda\phi'(b)} - \frac{e^{i\lambda\phi(a)}}{i\lambda\phi'(a)} - \int_a^b e^{i\lambda\phi(x)} \frac{d}{dx}\left(\frac{1}{i\lambda\phi'(x)}\right)\,dx. \end{aligned}$$

So

$$|I(\lambda)| \le \frac{2}{\lambda} + \frac{1}{\lambda}\int_a^b \left|\frac{d}{dx}\frac{1}{\phi'(x)}\right|\,dx$$

Since ϕ' is monotonic,

$$\begin{aligned} &= \frac{2}{\lambda} + \frac{1}{\lambda}\left|\int_a^b \frac{d}{dx}\frac{1}{\phi'(x)}\,dx\right| \\ &= \frac{2}{\lambda} + \frac{1}{\lambda}\underbrace{\left|\frac{1}{\phi'(b)} - \frac{1}{\phi'(a)}\right|}_{\le 1} \\ &\le \frac{3}{\lambda}. \end{aligned}$$

So $c_1 = 3$.

For the inductive step, assume the claim holds for some $k \ge 1$. Assume $|\phi^{(k+1)}(x)| \ge 1$ for all $x \in [a, b]$. Replacing ϕ by $-\phi$ if necessary, we may assume that $\phi^{(k+1)}(x) \ge 1$ for all $x \in [a, b]$. So $\phi^{(k)}$ is increasing on $[a, b]$. Then there exists at most one point $c \in [a, b]$ such that $\phi^{(k)}(c) = 0$. We have two cases:

1. $\exists c \in [a, b]$ such that $\phi^{(k)}(c) = 0$: Since $\phi^{(k)}$ grows at least linearly, there is a δ such that $|\phi^{(k)}(x)| \ge \delta$ for all $x \in [a, b] \setminus (c - \delta, c + \delta)$. Then

$$\left|\frac{d^k}{dx^k}\phi(\delta^{-1/k}x)\right| \ge 1 \qquad \forall \delta^{-1/k}x \in [a, b] \setminus (c - \delta, c + \delta).$$

   Then

$$I(\lambda) = \int_a^{c-\delta} e^{i\lambda\phi(x)}\,dx + \int_{c-\delta}^{c+\delta} e^{i\lambda\phi(x)}\,dx + \int_{c+\delta}^b e^{i\lambda\phi(x)}\,dx.$$

Using the change of variables $x = \delta^{-1/k} y$,

$$\left|\int_a^{c-\delta}\right| e^{i\lambda\phi(x)}\,dx = \left|\int_{\delta^{1/k}a}^{\delta^{1/k}(c-\delta)} e^{i\lambda\phi(\delta^{-1/k}y)}\delta^{-1/k}\,dy\right| \leq c_k(\delta\lambda)^{-1/k}$$

by the inductive hypothesis. Similarly,

$$\left|\int_{c+\delta}^{b} e^{i\lambda\phi(x)}\,dx\right| \leq c_k(\lambda\delta^{-1/k}.$$

For the remaining term, we have

$$\left|\int_{c-\delta}^{c+\delta} e^{i\lambda\phi(x)}\,dx\right| \leq 2\delta.$$

We get

$$I(\lambda)| \leq 2c_k(\lambda\delta)^{-1/k} + 2\delta.$$

Now choose δ such that $(\lambda\delta)^{-1/k} = \delta \implies \delta = \lambda^{-1/(k+1)}$. Then

$$|I(\lambda)| \leq \underbrace{2(c_k+1)}_{c_{k+1}}\lambda^{-1/(k+1)}.$$

2. $\phi^{(k)}(x) \neq 0$ for all $x \in [a,b]$: If $\phi^{(k)}(a) > 0$,

$$|I(\lambda)| \leq \left|\int_a^{a+\delta} e^{i\lambda\phi(x)}\,dx\right| + \left|\int_{a+\delta}^{b} e^{i\lambda\phi(x)}\,dx\right| \leq \delta + c_k(\lambda\delta)^{-1/k}$$

as in the previous case. Setting $\delta = \lambda^{-1/(k+1)}$, we get

$$|I(\lambda)| \leq (c_k+1)\lambda^{-1/(k+1)}.$$

Similarly, if $\phi^{(k)}(a) < 0$, then $\phi^{(k)}(b) < 0$. So we split

$$|I(\lambda)| \leq \left|\int_a^{b-\delta} e^{-\lambda\phi(x)}\,dx\right| + \left|\int_{b-\delta}^{b} e^{i\lambda\phi(x)}\,dx\right| \leq c_k(\delta\delta)^{-1/k} + \delta \leq (c_k+1)\delta^{-1/(k+1)}.$$

□

Corollary 1.1. *Let $\phi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{C}$ be smooth. Fix $k \geq 1$, and assume $|\phi^{(k)}(x)| \geq 1$ for all $x \in [a,b]$. If $k = 1$, assume also that ϕ' is monotonic. Then*

$$I(\lambda) = \int_a^b e^{i\lambda\phi(x)}\psi(x)\,dx$$

satisfies

$$|I(\lambda)| \leq c_k\lambda^{-1/k}\left[|\psi(b)| + \int_a^b |\psi'(x)|\,dx\right].$$

Proof. Write

$$I(\lambda) = \int_a^b \psi(x) \left(\frac{d}{dx} \int_a^x e^{i\lambda\phi(y)}\, dy \right) dx$$

Using integration by parts,

$$= \psi(b) \int_a^b e^{i\lambda\phi(y)}\, dy - \int_a^b \psi'(x) \cdot \left(\int_a^x e^{i\lambda\phi(y)}\, dy \right) dx.$$

□